Calculation of Separation Bubbles Using Boundary-Layer-Type Equations

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A simultaneous iteration approach to calculate laminar, incompressible viscous layers interacting with an inviscid outer flow is described. In the viscous region, the partially parabolized Navier-Stokes equations for laminar incompressible flows are solved iteratively using vertical line relaxation for the elliptic stream-function equation coupled with a marching procedure for the parabolic vorticity equation. In the inviscid region the vorticity is assumed to vanish and the stream-function equation is relaxed simultaneously with the viscous equations. The no-slip boundary conditions are expressed in terms of the stream function and are implemented implicitly leading to fast convergence. Examples of separated flows in a diffuser and in the neighborhood of a trailing edge are presented, and the effects of the Reynolds number and the geometry on the separation bubbles are discussed.

Introduction

FOR high Reynolds number flows, the viscous effects can be calculated in terms of a displacement thickness, that modifies the boundary condition for the inviscid problem. Coupling procedures are discussed in Ref. 1. On the other hand, the simultaneous iteration (SI) approach,[2-5] where both the viscous and inviscid equations are solved together, is very attractive since the coupling is implicit. In the present work, the flow is decomposed into two zones; the parabolized Navier-Stokes equations are solved in the viscous region simultaneously with the inviscid stream-function equation in the outer zone. Such a patching procedure is obviously simplified if the viscous equations are written in terms of the stream function and the vorticity. (The parabolized Navier-Stokes equations reduce automatically to the inviscid equation if the vorticity vanishes.) The location of the interface (or the patching line) is chosen outside the viscous layer. Hence, across the interface, the stream function, velocity, and vorticity are continuous.

The choice of the grid is determined by the resolution requirements; for example, in the viscous region the distribution of the grid points should be dense enough to capture the details of the boundary layer. The grid in the inviscid region, however, is much coarser, but, since a patching procedure is used, some restrictions may exist (at least in one direction). The same restrictions occur, of course, in solving the parabolized Navier-Stokes equations everywhere. The advantage of the present method is simply to use existing techniques for simulating inviscid and viscous flows and at the same time eliminate the coupling problems occurring in most viscous/inviscid interaction calculations. For example, the elliptic stream-function equation, which is valid in both regions, can be solved efficiently using successive line overrelaxation, while the parabolic vorticity equation in the viscous layer can be marched (in space). The stream-function and vorticity equations are coupled through the no-slip boundary condition. It is well known that explicit implementation of this condition leads to slow convergence or even divergence, hence it is critical to impose such a condition implicitly. This is done by solving a block (2×2) tridiagonal system in the viscous layer adjacent to a solid surface and implicitly enforcing the boundary conditions on the stream function, thus eliminating any explicit boundary condition on the vorticity. In the wake, however, the two equations may be decoupled (if the nonlinear interaction is lagged).

The viscous effects are manifested through the vorticity term in the stream-function equation. Due to this nonhomogeneous term, the stream function is displaced, and, consequently, the response of the inviscid flow over an effective body determines the surface pressure distribution. Since the inviscid and viscous flows are calculated simultaneously, no difficulty is expected in marching the vorticity equation provided proper upwind differences are used for the convective term. Also, the upstream influence in the viscous layer is allowed through the stream-function equation. The formulation does not depend on the boundary-layer approximation where the variation of the pressure across the boundary layer is assumed negligible. Nevertheless, the parabolic nature of the vorticity equation permits a marching procedure similar to the boundary-layer calculations.

The present method can be considered as marching in (artificial) time in the inviscid region while marching in space in the viscous region. Solving the parabolized Navier-Stokes equations by marching in time is documented in many references.[6-8] Methods based on marching in space have been recommended by some authors.[9-11] As noticed by Rubin and Reddy,[9] multiple global sweeps are needed to guarantee convergence. This is obvious since it is not conceivable to solve an elliptic equation by a single marching sweep. The difficulty Rubin and Reddy faced was to construct a relaxation procedure (multiple sweeps) for the inviscid (Euler) equations in the primitive variables. This difficulty is avoided in the present work since the standard line relaxation procedure can be used for the second-order elliptic stream-function equation. Even if the vorticity does not vanish in the inviscid

Presented as Paper 84-1585 at the AIAA 17th Fluid Dynamics, Plasma Dynamics and Lasers Conference, Snowmass, CO, June 25-27, 1984; received Nov. 21, 1984; revision received Aug. 2, 1985.

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region, an inviscid vorticity transport equation may be used there. For example, in the case of incompressible two-dimensional flows, ω is constant along a streamline. Moreover, a further simplification is achieved by restricting the viscous equation to a boundary layer, thus avoiding the general convection-diffusion problem.

The present method is at least applicable for flows simulated by viscous/inviscid interaction procedures based on boundary-layer and potential equations as well as for more complicated flows where existing coupling procedures are not efficient.

In the following, the formulation in terms of the stream function and vorticity is given. The numerical procedure is outlined and results are presented for internal flows in a diffuser as well as external flows around the trailing edge. Finally, the effects of Reynolds number and geometry on laminar separation in a diffuser and around a trailing edge are discussed.

Parabolized Navier-Stokes Equations

It is well known that the Navier-Stokes equations for an incompressible laminar flow over a body admit inner and outer expansions, in terms of a small parameter $\epsilon = Re^{-1/2}$, where the lowest-order equations describe the classical boundary-layer and inviscid irrotational flows, respectively. There is, however, a simple composite set of equations that contain both limits, namely, the parabolized Navier-Stokes equations. In Cartesian coordinates, the full equations are

$$u_x + v_y = 0$$

$$uu_x + vu_y = -(p_x/\rho) + \nu(u_{yy} + u_{xx})$$

$$uv_x + vv_y = -(p_y/\rho) + \nu(v_{yy} + v_{xx}) \qquad (1)$$

Neglecting the axial diffusion terms in Eqs. (1), the resulting set of equations is not really parabolic, hence, it is sometimes referred to as partially parabolized equations. Across a thin viscous region ($\mathcal{O}(\epsilon)$), the $\partial p/\partial y$ term is small and the boundary-layer equations are recovered. [According to Eqs. (1), at the wall $p_y = p_{yy} = p_{yyy} = 0$.] On the other hand, Eqs. (1) contain the inviscid flow equations as well.

The terms neglected in the full equations are higher order and the simplified equations are a good approximation, valid for describing many multiscale phenomena. For example, they contain at least all of the ingredients of triple-deck, first- and second-order theories (provided that the appropriate scales are used).

If the higher-order term v_{yy} is included, Eqs. (1) can be written in terms of the stream function ψ and vorticity ω and, hence, the pressure terms are eliminated:

$$u\omega_x + v\omega_y = \nu(\omega_{yy} + \omega_{xx}) \qquad (2a)$$

$$\psi_{xx} + \psi_{yy} = -\omega \qquad (2b)$$

In the viscous layer, the curvature ψ_{xx} is usually negligible, and Eqs. (2) reduce to a fourth-order equation in ψ with y as an independent variable, which can also be obtained by differentiating the x-momentum equation with respect to y, assuming p_{xy} is small.

Table 1 Present patching procedure

Inviscid	$\psi_{xx} + \psi_{yy} = 0$
	patching line
Viscous	$\psi_{xx} + \psi_{yy} = -\omega$
	$u\omega_x + v\omega_y = \nu\omega_{yy}$

Outside the viscous layer, the flow is usually irrotational and ω vanishes there. In case the inviscid flow is rotational, Eq. (2a) reduces to

$$u\omega_x + v\omega_y = 0 \qquad (2b')$$

and hence ω is constant along a streamline.

One advantage of the stream-function formulation is that the irrotationality condition can be implemented simply by forcing ω to vanish identically and thus the system reduces to one Laplace equation. Moreover, in the general case, Eqs. (2) require a 2×2 block tridiagonal system while Eqs. (1) call for a 3×3 block system.

Boundary Conditions

The outer boundary of the inviscid region is given in terms of the asymptotic behavior of the solution in the far field, or simply by requiring the flow to be uniform, i.e., $u = u_\infty$ and $\omega = 0$.

In the viscous layer and at the solid surface the no-slip boundary condition is imposed, or

$$\psi(x,0) = \text{const} \qquad (3a)$$

$$\psi_y(x,0) = 0 \qquad (3b)$$

At the left boundary $x = x_0$, ψ and ω are prescribed,

$$\psi = \psi(x_0,y) \qquad (3c)$$

$$\omega = \omega(x_0,y) \qquad (3d)$$

While at the right boundary, $x = L$, the boundary-layer equations are assumed to be valid, i.e.,

$$\psi_{yy} = -\omega \qquad (3e)$$

$$u\omega_x + v\omega_y = \nu\omega_{yy} \qquad (3f)$$

If L is very large, a similarity solution may be used as a special case of the boundary conditions (3e) and (3f).

Present Method

For steady, incompressible, laminar flows, parabolized as well as full Navier-Stokes equations are solved in many references. For example, in Ref. 12 a factored ADI method is used to calculate the flow over a family of blunt bodies; axisymmetric and turbulent flows are considered. In many problems, it is neither required nor recommended to solve the full equations everywhere in the field. A patching procedure is presented here, where the parabolized equations are used only in the viscous region, while in the outer region the viscous terms are switched off. If the inviscid flow is also irrotational, the vorticity equation drops automatically and the flow is represented by the stream-function equation only, as shown in Table 1.

The solution is independent of the position of the patching line. To demonstrate that fact, the skin friction coefficent was plotted for three different locations of the patching line in Fig. 1. As seen from that figure, the results were the same for all three cases. Also, the stream function, velocity, and vorticity should be continuous across such a line. This requirement can be satisfied if the patching line is chosen outside the viscous layer. Such a priori information is not a severe limitation of the method since the patching line can be adjusted during iteration. It should be located as close as possible to the edge of the viscous layer to take advantage of the inviscid flow model in the outer region. Basically, the method is an alternative numerical implementation of the boundary-layer concept. However, since patching procedure with the same grid and scales is used in the vicinity of the patching line, the boundary-layer equations are replaced by composite equations (where the inviscid model is a subset of it); hence, consistency is automatically guaranteed, i.e., both viscous and inviscid equations are valid on both sides of the patching line. The composite equations contain the boundary-layer equations as well as the normal-momentum equation. Thus, the present method is not really restricted to boundary-layer flows, where the pressure does not vary across a thin region. To demonstrate the details of the analysis, the flow in a symmetric diffuser is considered. Following Inoue[13] and Hoffman,[14] a shear transformation is used to reduce the physical domain into a rectangle in the computational plane. Let the wall shape be given by $Y_w(x)$. The new coordinate system (ξ,η) is (see Fig. 2):

$$\xi = x \tag{4a}$$

$$\eta = (Y - H_c)/H \tag{4b}$$

where $H = Y_w(x) - H_c$, and the viscous equations become

$$u\left(\omega_\xi - \frac{H'\eta}{H}\omega_\eta\right) + \frac{v}{H}\omega_\eta = \frac{1}{Re}\left(\frac{1+H'^2\eta^2}{H^2}\omega_{\eta\eta} + \frac{2H'^2 - HH''}{H^2}\eta\omega_\eta\right) \tag{5a}$$

$$\omega + \psi_{\xi\xi} + \frac{1+H'^2\eta^2}{H^2}\psi_{\eta\eta} - \frac{2H'\eta}{H}\psi_{\xi\eta} + \frac{2H'^2 - HH''}{H^2}\eta\psi_\eta = 0 \tag{5b}$$

where $Re = U_\infty L/\nu$.

The velocity components u and v are written in terms of ψ as

$$u = \frac{1}{H}\psi_\eta, \qquad v = -\psi_\xi + \frac{H'\eta}{H}\psi_\eta \tag{5c}$$

The boundary conditions are

$$u = v = 0 \qquad \text{at } \eta = 1 \tag{6}$$

and at $\xi = \xi_1$, ω and ψ are given by Blasius' boundary-layer solution, while at $\xi = \xi_2$, $\psi_{\xi\xi}$ is omitted from Eq. (5b).

For the inviscid flow, the governing equation is the same as Eq. (5b) with $\omega = 0$, and the boundary conditions are

$$\psi = \psi_c \qquad \text{at } \eta = 0 \tag{7}$$

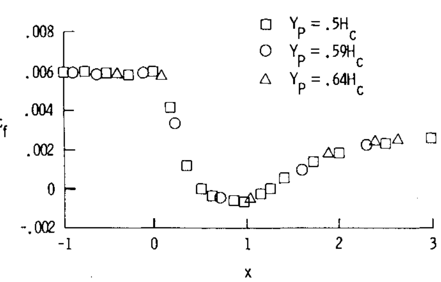

Fig. 1 Effect of the patching line location on the solution.

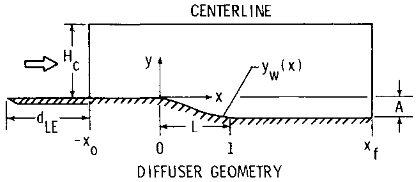

Fig. 2 Diffuser geometry.

and at $\xi = \xi_1$, $1/H\ (\partial\psi/\partial\eta) = u_c$, while at $\xi = \xi_2$, $\psi_{\xi\xi}$ is set equal to zero in Eq. (5b).

Finite difference approximations of Eqs. (5a) and (5b) lead to an algebraic system of equations that is solved iteratively. More precisely, central differences are used everywhere except for the term $u\omega_\xi$, which is upwind-differenced (backward or forward, depending on $u>0$ or $u<0$). (An improved discrete system can be obtained if conservative higher-order difference schemes are used.)

At each vertical line $(i=I)$, the correction equations have the following general form based on a Newton linearization procedure:

$$A\delta\psi_{I,j-1} + B\delta\psi_{I,j} + C\delta\psi_{I,j+1} + D\delta\omega_{I,j-1} + E\delta\omega_{I,j} + F\delta\omega_{I,j+1} = R_1 \tag{8a}$$

$$a\delta\psi_{I,j-1} + b\delta\psi_{I,j} + c\delta\psi_{I,j+1} + d\delta\omega_{I,j-1} + e\delta\omega_{I,j} + f\delta\omega_{I,j+1} = R_2 \tag{8b}$$

where R_2 is a function of ψ_{I+1}, ψ_I, and ψ_{I-1}. The equations were solved from upstream to downstream by a block (2×2) tridiagonal Thomas algorithm where the coefficients are calculated starting with the outer inviscid boundary conditions in the forward pass. The wall boundary conditions are enforced following Werle and Bernstein[7] using three-point backward difference representation of ψ_η, and the backward substitution step provides the solution for the correction equations. Since the ω and ψ equations are coupled, both boundary conditions $u=v=0$ can be imposed in terms of ψ with no explicit boundary condition on ω.

Equation (8a) reduces to a tridiagonal system in the inviscid region, hence a special Gaussian elimination procedure that accounts for variable bandwidth could be more efficient.

To improve the convergence, the vorticity equation is augmented by an artificial time-dependent term proportional to $\omega_t\Delta t$, while an overrelaxation parameter is used to relax the stream-function equation. A local iteration at each line

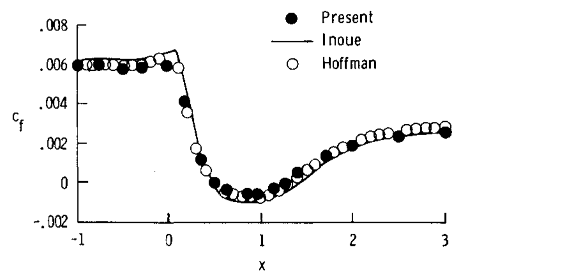

Fig. 3 Skin friction distribution for the diffuser (separated flows).

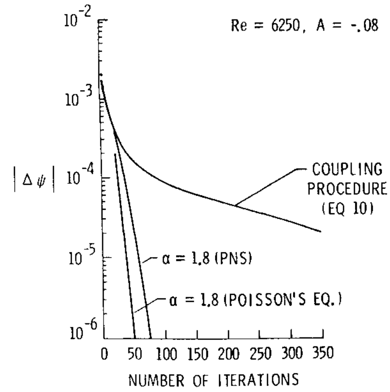

Fig. 4 Convergence history of the present method.

helps the nonlinear interaction between ω and ψ in the viscous region, while between marching sweeps a global solver for ψ (assuming ω is frozen) helps the spread of the viscous effects. The latter possibility has not been tested in the present study. The results are compared with those of Inoue[13] and Hoffman.[14] While Hoffman solved the parabolized Navier-Stokes equations everywhere, Inoue used viscous/inviscid interaction with a direct coupling procedure in terms of a displacement thickness.

Numerical Results

Separated Flows in a Diffuser

Calculations of flows in a divergent duct for different Reynolds numbers and expansion ratios are presented for the following geometry (see Fig. 2):

$$\begin{aligned} Y_w(x) &= 0.0 \qquad && -1.0 \le x \le 0.0 \\ &= Ax^2(3-2x) && 0.0 \le x \le 1.0 \\ &= A && 1.0 \le x \le 3.45 \end{aligned} \tag{9}$$

The skin friction distributions obtained by the present patching procedure are shown in Fig. 3. The present results of the viscous/inviscid interaction procedure are in good agreement with those of Refs. 13 and 14. The convergence rate of the present method is demonstrated in Fig. 4 and is compared to the results of a coupling procedure where the viscous and inviscid equations are solved separately assuming that the value of ψ at the patching line is known. This value is updated using the following formula:

$$\psi_p^{n+1} = \psi_p^n + \alpha(u_I - u_v) \tag{10}$$

where

$$u_I \simeq (\psi_{p+1}^n - \psi_p^n)/\Delta y$$

$$u_v \simeq (\psi_p^n - \psi_{p-1}^n)/\Delta y$$

α is a relaxation parameter and ψ_{p+1}^n and ψ_{p-1}^n are obtained from the inviscid and viscous calculations, respectively. It is clear that the convergence of the simultaneous iteration procedure is better. If the vorticity distribution is known, the stream-function equation (Poisson's equation) is solved with a line overrelaxation procedure and the rate of convergence is plotted in Fig. 4. The present method of solving the partially parabolized Navier-Stokes (PPNS) system is almost as fast as solving one Poisson equation on the same grid. To see the effect of applying the PPNS equations across the entire computational region, a numerical experiment was run for a Reynolds number of 6250 and the diffuser parameter A of -0.08 by executing the computations twice; once using the PPNS equations across the whole domain, and the second time the vorticity equation was dropped out in the inviscid region. That resulted in a 10% saving of CPU time. Of course the saving in the computational time is a function of the size of the inviscid region. For the case under consideration, the size of the inviscid region is relatively small compared to external flow problems.

A detailed study about the effect of the different parameters (Reynolds number, inlet diameter, and mean slope of the diffuser wall) on the flow separation inside the diffuser is depicted in Ref. 24.

Trailing-Edge Problems

The classical boundary-layer theory breaks down at the trailing edge, since a singular solution due to the discontinuous boundary condition is admitted as shown by Goldstein.[15] Stewartson[16] and Messiter[17] analyzed laminar attached flows on a semi-infinite flat plate using the triple-deck theory. The bottom deck equations are the same as boundary-layer equations (if the correct scales are used), the middle deck is a shear flow, and the outer deck is the inviscid irrotational flow. Due to the viscous/inviscid interaction, the effects of the discontinuous boundary condition are limited and the pressure gradient is large but localized. There is, however, a small region where the full Navier-Stokes equations are needed in the immediate neighborhood of the trailing edge, however, such a refinement leads to very local modifications.

The results of the triple-deck theory compare favorably to those of the interacting boundary-layer calculations as well as to the parabolized Navier-Stokes solutions (see, for example, Refs. 18-20).

Good agreement between the asymptotic theory and the interacting boundary-layer calculations is obtained for high Reynolds numbers. On the other hand, the interacting boundary-layer results seem to agree with the solution of the composite (PPNS) equations in a wider range. The latter is more general since the pressure variations across the viscous layer are allowed. At least for turbulent flows, a large pressure gradient at the trailing edge has been observed both

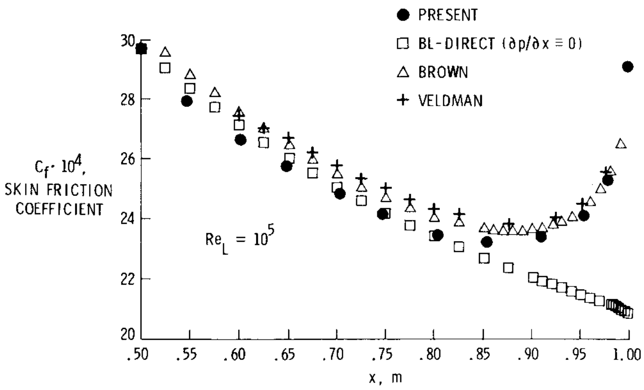

Fig. 5a Skin friction distribution for the flat plate.

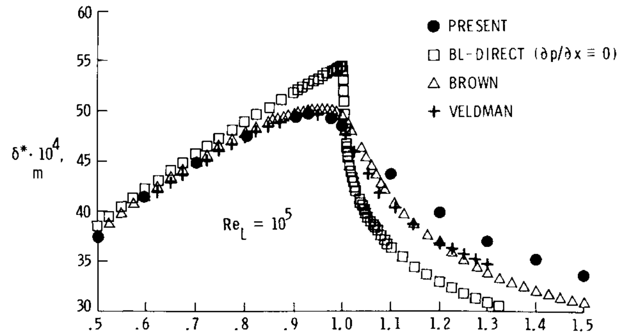

Fig. 5b Displacement thickness distribution for the flat plate.

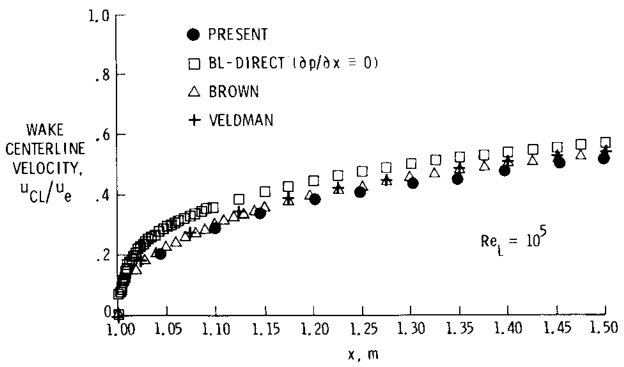

Fig. 5c Wake centerline velocity distribution for the flat plate.

experimentally and in the numerical solution of Navier-Stokes equations (with different turbulence models).

Calculations of attached flows in the trailing-edge region of a flat plate based on the present patching procedure are in good agreement with the results of Veldman[18] and Brown,[10] as shown in Fig. 5. As seen from Fig. 5b, there is a small difference between the present results and those predicted by the interacting boundary layer (IBL) method in the vicinity of the trailing edge. Such difference between the PPNS and IBL methods were reported by Rubin and Reddy in Ref. 25. As expected, these differences will die out far downstream as shown in Fig. 6. In these calculations, Cartesian grids are used with uniform irrotational flow at the outer boundary ($\psi_y = U_\infty$ and $\omega = 0$). The no-slip boundary conditions are forced on the plate surface.

No special treatment of the wake is needed for the general case as described in Fig. 7. Unlike many boundary-layer calculations, no assumption regarding the wake centerline is required. For the symmetric case, however, the conditions ψ and $\omega = 0$ are used at the axis.

For a wedged trailing edge, the flow may separate (even inviscid shear flow may have closed streamlines, as in Kuchemann's model.[21]). The theory for high Reynolds number laminar flows is discussed in Ref. 23. The present method, without any modification, is still applicable. The present results for the trailing-edge problem compared with the solution obtained by Vatsa[26] are shown in Fig. 8. Figure 8b shows the skin friction distribution for the boattail (the corresponding solution for the trailing edge is identical until $x = 1$). In these calculations, the geometry of Eq. (9) is used for a trailing-edge and a boattail thickness distribution. Both viscous and inviscid equations are relaxed simultaneously leading to fast convergence even for separated flows. The trailing-edge and the boattail streamline patterns for a Reynolds number of 10^5 is shown in Fig. 9. As expected, the bubble for the trailing-edge problem is only slightly larger than that for the boattail.

A stretched grid is used to capture the details of the viscous region in the vicinity of the trailing edge, accounting for the proper scales of the triple-deck theory. No difficulties occur with a smooth (cusped) distribution. For a trailing edge with a finite angle, a shear transformation produces discontinuous metrics and a better transformation is needed. For example, local polar coordinates call for the full elliptic equations, while the parabolized set of equations may be adequate if a slit transformation is used.

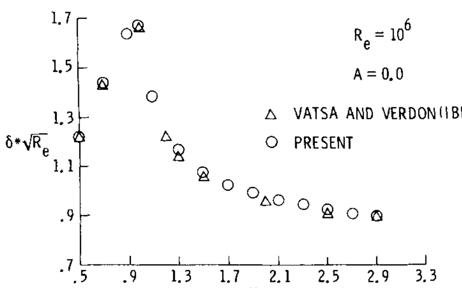

Fig. 6 Effect of the downstream boundary location.

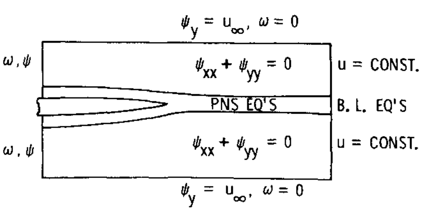

Fig. 7 Trailing-edge problem.

Concluding Remarks

A patching method for solving viscous/inviscid interaction problems is presented and its advantages are discussed. It seems that at least for steady, two-dimensional, incompressible laminar flows, the use of the stream function for both viscous and inviscid flows is useful; the resulting governing equations are amenable for standard numerical analysis. For boundary-layer-type flows, the vorticity equations can be solved by marching in the streamwise direction while a successive line overrelaxation is applied to the stream-function equations. The combination seems more natural than the classical alternating-direction-implicit method.

Simulations of separated flows in diffusers and in the neighborhood of trailing edges are presented. The present method is applicable, as long as the separation bubble is closed and confined to a small region. As expected, there are some difficulties if the bubble is not closed.

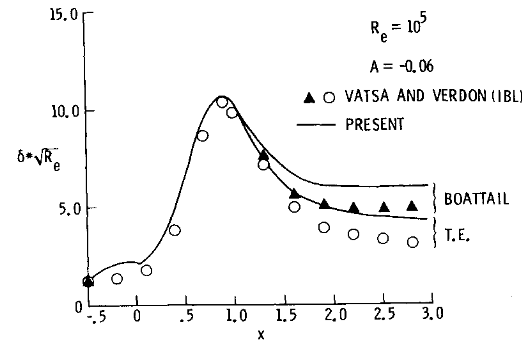

Fig. 8a Displacement thickness distribution for the boattail and trailing edge.

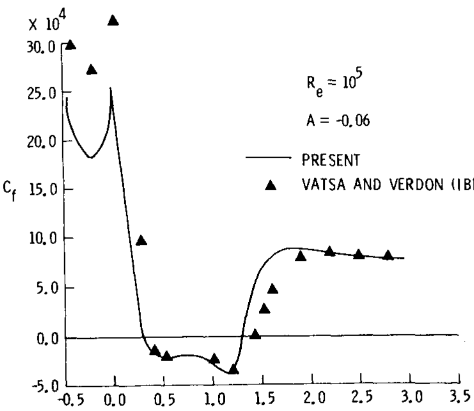

Fig. 8b Skin friction distribution for the boattail.

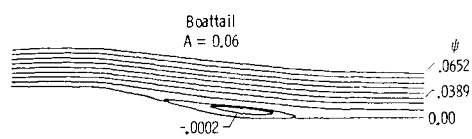

Fig. 9a Streamlines pattern for the boattail.

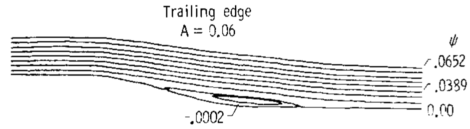

Fig. 9b Streamlines pattern for the trailing edge.

Acknowledgments

The authors would like to acknowledge the support of NASA Langley Research Center, Office of Naval Research, Contract N00014-80-C-494 and the U.S. Air Force Office of Scientific Research, Contract F49620-84-C-0032. Also, the authors would like to thank Dr. V. Vatsa of NASA Langley Research Center for providing his results in Figs. 6 and 8.

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